SUBPOLYTOPES OF STACK POLYTOPES

BY

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ABSTRACT

A polytope P is called a *subpolytope* of a polytope Q if vert $P \subseteq$ vert Q. The purpose of this paper is to construct examples of 3-dimensional polytopes which are not subpolytopes of stack polytopes. Previously no such examples were known.

1.

Let P be an n-polytope† in R^n, F be one of its facets, and p be a point beyond F and beneath all the other facets of P. Then the polytope conv$(\{p\} \cup P)$ is said to arise from P by *adjoining the pyramid* conv$(\{p\} \cup F)$ *to the facet* F *of* P. Any polytope which can be obtained from a simplex by repeatedly applying this process of adjoining pyramids is called a *stack polytope* and the class of all n-dimensional stack polytopes is denoted by $\mathscr{S}^n$. The simplexes that are adjoined in the construction of an $S \in \mathscr{S}^n$, together with the initial simplex, are known as the *components* of S. It is clear that for $n \geqq 3$ any vertex of S of valency n belongs to precisely one component of S. Deleting this component and applying an obvious inductive argument on the number of components enables us to establish that the components of S are uniquely determined by S itself. Moreover, similar considerations lead us to the conclusion that any polytope combinatorially equivalent to a stack polytope must also be a stack polytope.

Interest in the class $\mathscr{S}^n$ arises from the fact that stack polytopes have proved useful in several investigations, for example, in the proof of the *lower bound conjecture* for convex polytopes, and in the construction of polytopes which possess short edge-paths.

† For the properties of convex polytopes and an explanation of the standard terminology and notation used here, see B. Grünbaum, *Convex Polytopes*, London-New York-Sydney, 1967, or P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, London Mathematical Society Lecture Note Series Volume 3, Cambridge, 1971.

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A polytope P is called a *subpolytope* of a polytope Q if vert $P \subseteq$ vert Q, and we shall write $P \leqq Q$. The purpose of this paper is to consider polytopes which can occur as subpolytopes of stack polytopes. We shall restrict ourselves to dimension $n = 3$, not only because of the intuitive appeal of this case, but also because the properties we shall state generalise without difficulty to all dimensions $n > 3$. The case $n = 2$ is trivial.

The investigation to be described started with the following question, which was asked of the author by a well-known mathematician: *Does* $P \leqq Q \in \mathcal{S}^3$

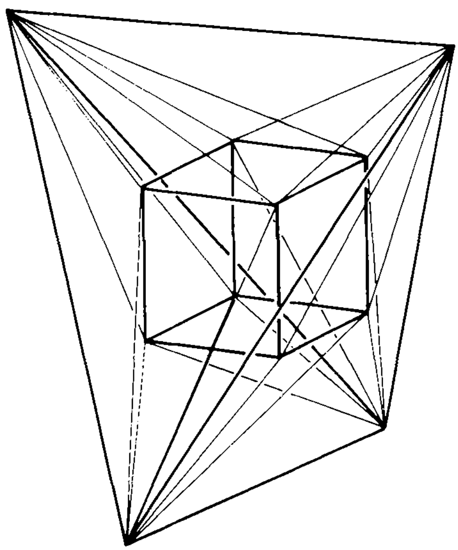

Fig. 1.

imply $P \in \mathcal{S}^3$? Clearly the answer to this question is in the negative if P is not simplicial. For example, a regular cube C is a subpolytope of a stack polytope which may be constructed in the following way. We place the cube so that its centre coincides with the centre of a regular tetrahedron T and so that two vertices of C lie beyond each of the four 2-faces of T (see Fig. 1). Then so long as no edge of C is parallel to an edge of T, conv (vert $C \cup$ vert T) is a stack polytope with nine components.

Even if we restrict attention to simplicial polytopes, the answer to the queştion is still negative, the simplest example being provided by the stack polytope with four components whose Schlegel diagram appears in Fig. 2. If we delete the ringed vertex, the convex hull of the remaining six vertices is an octahedron, which is not a stack polytope since it has no vertex of valency

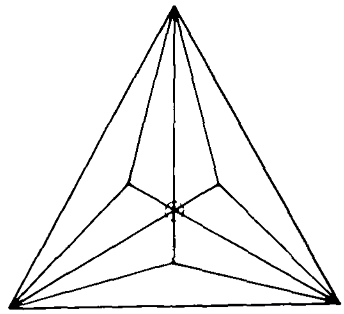

Fig. 2.

three. Examination of this and similar examples led the author to suppose, at one time, that every 3-polytope might be exhibited as a subpolytope of some $Q \in \mathcal{S}^3$. The purpose of this short note is to show that this is not so: we shall construct a simplicial 3-polytope which is not a subpolytope of *any* stack polytope.

2.

Let S^2 be the unit 2-sphere (boundary of the 3-ball B^3) centred at the origin $o \in R^3$. Let E be an ϵ-net on S^2, that is to say, a set of points on S^2 such that every point of S^2 lies within a distance ϵ of some point of E. Write $P(\epsilon) = \operatorname{conv} E$. Then we shall show that if ϵ is sufficiently small $P(\epsilon)$ is not a subpolytope of any $Q \in \mathcal{S}^3$. Indeed, let us suppose $P(\epsilon) \leqq Q \in \mathcal{S}^3$. Then we shall show that this assumption, for all $\epsilon > 0$, leads to a contradiction. This contradiction is the consequence of the following simple observations:

(i) There exists $\delta > 0$ such that $P(\epsilon) \supseteq (1-\delta)B^3$, and δ may be chosen so that $\delta \to 0$ as $\epsilon \to 0$. Let us suppose, therefore, without loss of generality, that $\delta < \frac{1}{4}$ and $\epsilon < \frac{1}{4}$.

(ii) For any such δ let x be *a* point outside B^3 so that $\operatorname{conv}(\{x\} \cup (1-\delta)B^3)$ cuts a disc of radius ϵ on S^2. This disc necessarily contains a point of E, so that $P \leqq Q$ implies that x cannot be a vertex of Q. Moreover, if $|x| = 1 + \zeta$, then as $\epsilon \to 0$, $\delta \to 0$, clearly $\zeta \to 0$ also.

(iii) Every point of $P(\epsilon)$ must lie in at least one component of Q. In particular, let T^3 denote the component of Q that contains the origin o.

(iv) Since every edge of T^3 is an edge of Q, it follows from (i) that every edge of T^3 must lie at a distance at least $1-\delta$ from o. From (ii) we deduce that every vertex of T^3 must lie at a distance at most $1+\zeta$ from o. These two facts imply that every edge of T^3 has length at most

$$l(\zeta,\delta)=[8(\zeta+\delta)+4(\zeta^2-\delta^2)]^{\frac{1}{2}},$$

and we note that $l(\zeta,\delta)\to 0$ as $\epsilon\to 0$.

(v) Choose ϵ so small that $l(\zeta,\delta)<\frac{1}{3}$, say. Then T^3 has diameter less than $\frac{2}{3}$, and as its vertices lie outside $(1-\delta)B^3\supseteq\frac{2}{3}B^3$, it cannot contain the point o.

This contradiction completes the proof of the assertion.

3.

It would be of some interest to determine the 'minimal polytope' (i.e. the one with the fewest vertices) that is not a subpolytope of any stack polytope. The above existence proof gives us little information. It is possible, however, by using numerical estimates of the various quantities to show that there is a 3-polytope with 32 vertices (constructed by adjoining a pentagonal pyramid to each facet of a regular dodecahedron) that is not a subpolytope of a stack polytope. Probably, though, the 'minimal polytope' has far fewer than 32 vertices.

An interesting problem is whether the result we have proved above holds in a combinatorial sense, that is, we ask: *Is every combinatorial type of n-polytope represented by some subpolytope of some n-dimensional stack polytope*? The author conjectures that the answer to this question is in the affirmative, at least in the case $n=3$.

In this connection we mention that Peter Kleinschmidt has recently shown that there exist 3-polytopes whose edge-graphs are not homeomorphic to subgraphs of edge-graphs of stack polytopes. His argument (private communication) is as follows. Suppose that the edge-graph of a stack polytpe Q possessed a subgraph homeomorphic to the edge-graph of an octahedron P. Let A, B; C, D; E, F be the three pairs of opposite vertices of P and A', B'; C', D'; E', F' be the corresponding vertices of Q. The fact that there are four mutually disjoint edge-paths joining A to B in P implies that there must be four mutually disjoint edge-paths joining A' to B' in Q. Now it is known† that every two vertices of a 3-dimensional stack polytope S are joined by exactly three mutually disjoint edge-paths unless the two vertices belong to the same edge of S. We deduce that A', B' are joined by an edge of Q, and similarly C', D' and E', F' are joined by edges. We deduce that the edge-graph of Q must contain a subgraph homeomorphic to the complete graph on six vertices, which is impossible. This contradiction shows that the original supposition was false, and so the assertion is proved.

† P. Kleinschmidt, Doctoral dissertation, Bochum, Germany, 1974.

Finally we remark that the question considered here is only one of a large class of similar problems, about whose solution very little is known. It would be interesting to consider classes other than $\mathscr{S}^n$ and to determine whether any of these have the property that *all* n-polytopes can be exhibited as subpolytopes of their members, either in the strict sense, or in the combinatorial sense mentioned above.

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